# Chapter 4 Index numbers and their uses

- **Percentage price index for a single item**
  - $I = \frac{P_n}{P_0} \times 100$

    $P_n$ : price at time period n

    $P_0$ : price at base time period

- **Aggregate price index**
  - Unweighted aggregate price index: $\frac{\sum_{i=1}^{N} P_i^n}{\sum_{i=1}^{N} P_i^0} \times 100$

    $P_i^n$ : price at time period n for item i (*items should be in the same unit*)

    $P_i^0$ : price at base time period for item i
  - Weighted aggregate price index:
    - Laspeyres Index: $\frac{\sum_{i=1}^{N} P_{n_i} q_{0_i}}{\sum_{i=1}^{N} P_{0_i} q_{0_i}} \times 100$
    - Paasche Index: $\frac{\sum_{i=1}^{N} P_{n_i} q_{n_i}}{\sum_{i=1}^{N} P_{0_i} q_{n_i}} \times 100$
    - Fixed-weight aggregate price index: $\frac{\sum_{i=1}^{N} P_{n_i} q_{a_i}}{\sum_{i=1}^{N} P_{0_i} q_{a_i}} \times 100$

      $q_0$ = quantity (weighting) at base period

      $q_n$ = quantity (weighting) at time period n

      $q_a$ = chosen quantity (weighting) e.g. from survey

- **Hang Seng Index (HSI):**

$$HSI_t = \frac{\sum_{i=1}^{50} p_i^t N_i^t}{\sum_{i=1}^{50} p_i^{t-1} N_i^{t-1}} \times HSI_{t-1}$$

$p_i^t$ is the closing price of stock *i* at time *t*

$N_i^t$ is the number of shares of stock *i* at time *t*

$p_i^t N_i^t$ is the capitalization of stock *i* at time *t*

We usually assume that the base year index = 100 for the Index mentioned above.

**Example 1**

The following table summarizes some historical data of three stocks. To better understand their performance, you can construct an index of these three stocks using a method similar to that of constructing Hang Seng Index. If the value of your index at Day 1 is 100, what are the values of the index at Day 2 and Day 3?

| | Price ($) | | | Number of Shares | | |
|---|---|---|---|---|---|---|
| | Day 1 | Day 2 | Day 3 | Day 1 | Day 2 | Day 3 |
| Stock A | 98 | 100 | 101.5 | 1,000,000 | 1,000,000 | 1,500,000 |
| Stock B | 10 | 5.2 | 5.3 | 1,000,000 | 1,300,000 | 2,200,000 |
| Stock C | 16 | 19 | 65 | 1,000,000 | 1,000,000 | 250,000 |

**Solution:**

Capital in Day 1 =

Similarly,

Capital in Day 2 =

As we want to construct an index similar to HIS, it should be:

$$HSI_t = \frac{\sum_{i=1}^{40} p_i^t N_i^t}{\sum_{i=1}^{40} p_i^{t-1} N_i^{t-1}} \times HSI_{t-1}$$

where base is Day 1 and index is 100.

Therefore,

Index of day 2 =

Index of day 3 =

- **The market model:**

$$r_i = \alpha_i + \beta_i r_I + \varepsilon_i$$

where $r_i$ = return on **security** *i* for some given period

$r_I$ = return on **market index** *I* for some given period

So we need to estimate $\alpha_i$ and $\beta_i$

$$\hat{\beta}_i = \frac{\sum_{j=1}^{n}((r_i)_j - \bar{r}_i)((r_I)_j - \bar{r}_I)}{\sum_{j=1}^{n}((r_I)_j - \bar{r}_I)^2}$$

If we divide *(n-1)* on both the numerator and denominator, we get:

$$\hat{\beta}_i = \frac{\text{cov}(return \quad of \quad \text{sec}urity \quad i, return \quad of \quad market \quad index)}{\text{var}(return \quad of \quad market \quad index)} = \frac{\sigma_{iI}}{\sigma_I^2}$$

The slope, $\beta_i$ measures the sensitivity of the security's return to the market index's returns.

| | | |
|---|---|---|
| Aggressive stock | = | stock with $\beta_i > 1$ |
| Defensive stock | = | stock with $\beta_i < 1$ |
| Mirror stock | = | stock with $\beta_i = 1$ |

In most practice, $\beta_i > 0$. If $\beta_A > \beta_B$, it means that the returns of A are more sensitive than the returns of B to the returns of the market index.

- **Diversification**

From Market Model: $r_i = \alpha_i + \beta_i r_I + \varepsilon_i$

Take variance on both sides, $\sigma_i^2 = \beta_i^2 \sigma_I^2 + \sigma_{\varepsilon i}^2$

Total risk = market risk + unique risk

  - Portfolio total risk:

    $r_P = \alpha_P + \beta_P r_I + \varepsilon_P$

    Thus, $\sigma_P^2 = \beta_P^2 \sigma_I^2 + \sigma_{\varepsilon P}^2$, $\beta_P^2 = (\sum_{i=1}^{n} x_i \beta_i)^2$ and $\sigma_{\varepsilon P}^2 = \sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2$

  - Portfolio market risk:

    $\beta_P^2 \sigma_I^2 = (\sum_{i=1}^{n} x_i \beta_i)^2 \sigma_I^2$

    As $\beta_P$ is the linear combination of $\beta_i$, increase the number of stocks in the portfolio will not change the magnitude of portfolio market risk much.

  - Portfolio unique risk:

    $\sigma_{\varepsilon P}^2 = \sum_{i=1}^{n} x_i^2 \sigma_{\varepsilon i}^2$

    Let $x_i = 1/n$, $\sigma_{\varepsilon P}^2 = \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma_{\varepsilon i}^2}{n} \to 0$ as *n* increases

    Therefore, increase the number of stocks in the portfolio can diversify the portfolio unique risk.

**Example 2**

HSBC Holdings (HSBC) and Cheung Kong (CK) are two Hang Seng Index (HSI) constituent stocks. The price of each individual stock together with the HSI for the period from 1997 to 2000 are given as follows:

| | HSI | HSBC | CK |
|---|---|---|---|
| 12/31/97 | 10722 | 63.67 | 50.75 |
| 6/30/98 | 8543.1 | 63.17 | 38.1 |
| 12/31/98 | 10048.6 | 64.33 | 55.75 |
| 6/30/99 | 13532.1 | 94.33 | 69 |
| 12/31/99 | 16962.1 | 109 | 98.75 |
| 6/30/00 | 16155.8 | 89 | 86.25 |
| 12/29/00 | 15095.5 | 115.5 | 99.75 |
| 6/29/01 | 13042 | 92.25 | 85 |

a. Fit the market model to HSBC and CK respectively, and interpret the result (you can neglect the effect of dividend in your analysis).

b. Design a portfolio of HSBC and CK with minimum risk (you can neglect the effect of dividend in your analysis).

**Solution:**

First, we find the return in the period *I* using this formula:

$$R_i = \frac{D_i}{V_{i-1}} + \frac{V_i - V_{i-1}}{V_{i-1}}$$

| Period | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| Return of HSI | -0.2032 | 0.1762 | 0.3467 | 0.2535 | -0.0475 | -0.0656 | -0.1360 |
| Return of HSBC | -0.0079 | 0.0184 | 0.4663 | 0.1555 | -0.1835 | 0.2978 | -0.2013 |
| Return of CK | -0.2493 | 0.4633 | 0.2377 | 0.4312 | -0.1266 | 0.1565 | -0.1479 |

From the table above, we first calculate the expected return, SD and covariance.

a. For the market model of HSBC:

$$\hat{\beta} = \frac{\operatorname{cov}(R_{HSBC}, R_{HSI})}{\sigma^2_{HSI}} =$$

$$\hat{\alpha} = \overline{R_{HSBC}} - \hat{\beta}\overline{R_{HSI}} =$$

The market model of HSBC:

For the market model of CK:

$$\hat{\beta} = \frac{\operatorname{cov}(R_{CK}, R_{HSI})}{\sigma^2_{HSI}} =$$

$$\hat{\alpha} = \overline{R_{CK}} - \hat{\beta}\overline{R_{HSI}} =$$

The market model of CK:

HSBC is a stock with

CK is an stock with

b. Portfolio: $R_p = XR_{HSBC} + (1 - X)R_{CK}$

Using the formula: $X = \dfrac{\sigma^2_{CK} - \operatorname{cov}(R_{HSBC}, R_{CK})}{\sigma^2_{HSBC} + \sigma^2_{CK} - 2\operatorname{cov}(R_{HSBC}, R_{CK})}$,

The portfolio is:

Thus, the expected return of portfolio (with minimum risk):

Risk of the portfolio: