2. Duration

Macaulay's duration (D) is computed as the weighted average of the times for the payments made by the bond. The weighting of each time is the discounted value of the payment, which shows the importance of that payment. This is just a measure of effective maturity of that bond.

Modified duration (MD) is computed as -1 times the relative sensitivity, the relative sensitivity is calculated as the derivative of the bond price with respect to yield divided by the bond price.

$Duration is defined as absolute sensitivity, which can be calculated by the relative sensitivity times the bond price.

Modified duration and $Duration measures the relative change and absolute change of bond price in response to the small change in yield respectively.

Annual compounding

$$D = \frac{\sum_{i=1}^{m} \frac{t_i F_i}{(1+y)^{t_i}}}{V(y)}$$

$$MD = -\frac{V'(y)}{V(y)} = \frac{\sum_{i=1}^{m} \frac{t_i F_i}{(1+y)^{t_i+1}}}{V(y)} = \frac{D}{1+y}$$

$$\$\text{Duration} = V'(y) = -\sum_{i=1}^{m} \frac{t_i F_i}{(1+y)^{t_i+1}}$$

Continuous compounding

$$D = MD = -\frac{V'(y)}{V(y)} = \frac{\sum_{i=1}^{m} t_i F_i e^{-y^c t_i}}{V(y)}$$

$$\$\text{Duration} = V'(y) = -\sum_{i=1}^{m} t_i F_i e^{-y^c t_i}$$

3. Duration Hedging

Hedge ratio (q) is the number of hedging instrument you have to hold in order to hedge one unit of the original bond portfolio. Denote the bond portfolio as P and the hedging instrument as H.

$$dP + qdH = \big(P'(y) + qH'(y)\big)dy = 0$$

$$q = -\frac{P'(y)}{H'(y)} = \frac{-P \times MD_P}{H \times MD_H}$$

Example

An investor holds 100,000 units of a bond whose features are summarized in the following table. He wishes to be hedged against a rise in interest rates.

Maturity	Coupon rate	YTM	Face value
18 Years	9.5%	8%	$100

Characteristics of the hedging instrument, which is a bond here, are as follow:

Maturity	Coupon rate	YTM	Face value
20 Years	10%	8%	$100

Coupon frequency and compounding frequency are assumed to be annual.

a. Calculate the modified durations for the two bonds.

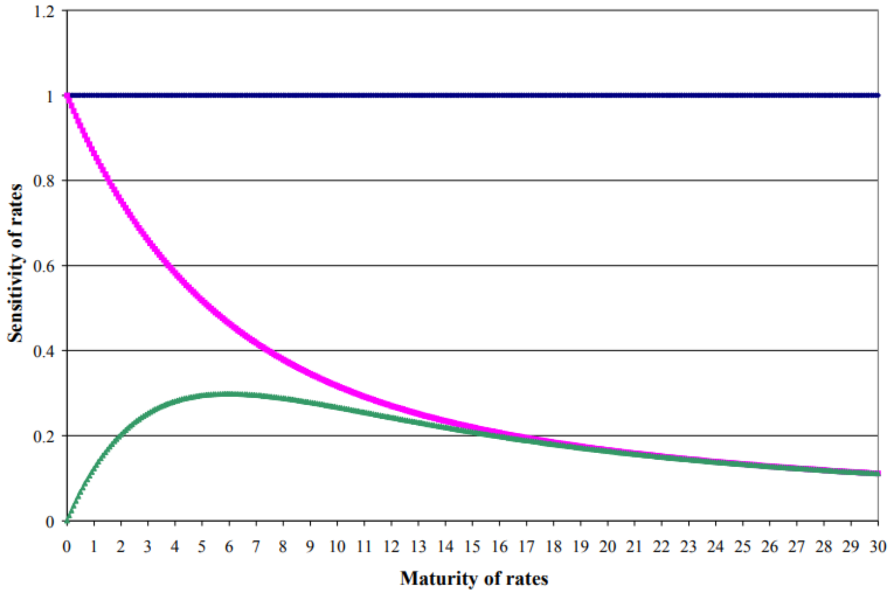
b. What is the quantity of the hedging instrument that the investor has to sell?

c. Suppose that the YTM increases instantaneously by 0.1%, what happens if the bond is not hedged? What happens if the bond is hedged as above?

4. Nelson Siegel Model

$$R_T^c = \beta_0 + \beta_1\left[\frac{1-\exp\left(-\frac{T}{\tau}\right)}{\frac{T}{\tau}}\right] + \beta_2\left[\frac{1-\exp\left(-\frac{T}{\tau}\right)}{\frac{T}{\tau}} - \exp\left(-\frac{T}{\tau}\right)\right]$$

β_0 is the long-term interest rate as β_0 is the limit of the rate when time goes to infinity. β_1 is the long-to-short-term spread. β_1 is the curvature parameter. τ is the time scaling parameter.



2018 midterm

1. At date $t=0$, the values of Nelson and Siegel parameters are as follows:

β_0	β_1	β_2	τ
3%	*-1%*	*6%*	*1*

Consider the following three bonds (coupon frequency is semi-annual):

Bond	Maturity	Coupon rate	Face value
1	0.5 year	4%	\$1000
2	1 year	3%	\$1000
3	1.5 year	2%	\$1000

a. (6%) Calculate the discount factor for 0.5 year, the discount factor for 1 year and the discount factor for 1.5 year based on the Nelson and Siegel model.

b. (9%) Calculate the sensitivities of a portfolio consisting of 3 unit of bond 1, 2 units of bond 2 and 1 unit of bond 3 to the betas.

c. (6%) Making use of the sensitivities calculated in part b., approximate the change in the value of the portfolio considered in part b. if there is an instantaneous change of the values of β_0, β_1 and β_2 to *3.5%*, *-1.5%* and *6.5%* respectively. The value of τ remains unchanged.

d. (6%) Suppose you are told that the term structure at $t=0$ exhibit a humped yield curve shape. What is the effect on the term structure if β_0 increases while the other parameters are kept unchanged? What is the effect on the term structure if β_2 increases while the other parameters are kept unchanged?